Quasinormal behavior of the D-dimensional Schwarzshild black hole and higher order WKB approach

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Abstract

We study characteristic (quasinormal) modes of a D-dimensional Schwarzshild black hole. It proves out that the real parts of the complex quasinormal modes, representing the real oscillation frequencies, are proportional to the product of the number of dimensions and inverse horizon radius $\sim D\ r_0^{-1}$. The asymptotic formula for large multipole number l and arbitrary D is derived. In addition the WKB formula for computing QN modes, developed to the 3th order beyond the eikonal approximation, is extended to the 6th order here. This gives us an accurate and economic way to compute quasinormal frequencies.

1 Introduction

Within the framework of the brane world models the size of extra spatial dimensions may be much larger than the Plank's length, and the fundamental quantum gravity scale may be very low ($\sim$ Tev). When considering models with large extra dimensions the black hole mass may be of order Tev., i.e. much smaller than the Plank's mass. There is a possibility of production of such mini black holes in particle collisions in colliders and in cosmic ray experiments [1]. Estimations show that these higher dimensional black holes can be described by classical solutions of vacuum Einstein equations. Thus the investigation of general properties of these black holes, including perturbations and decay of different fields around them, attracts considerable interest now (see for example [2], [3] and references therein).

It is well-known that when perturbing black hole it undergoes damping oscillations which are characterized by some complex eigenvalues of the wave equations called *quasinormal frequencies*. Their real parts represent the oscillation frequencies, while the imaginary ones determine the damping rates of the modes. The quasinormal modes (QN) of black holes (BH's) depend only on a black hole parameters and not on a way in which they were excited. QN's are called, therefore, "footprints" of a black hole. Being a useful characteristic of black hole's dynamics, quasinormal modes are studied also within different contexts now: in Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence (see for example [4]- [15] and references therein), because of the possibility of observing quasinormal ringing of astrophysical BH's (see [16] for a review), when considering thermodynamic properties of black holes in loop quantum gravity [17]-[20], in the context of possible connection with critical collapse [4], [9], [21], [22].

Thus it would be interesting to know, from different grounds, what happen with QN spectrum a black hole living in D -dimensional space-time [23], [3]. The present paper is two-fold: First we extend the WKB method of Schutz, Will and Iyer for computing QN modes from the 3th to the 6th order beyond the eikonal approximation (see Sec.II and Appendix I). In a lot of physical situations this allows us to compute the QNMs accurately and quickly without resorting to complicated numerical methods. In Appendix II QN modes of $D = 4$ Schwarzshild black hole induced by perturbations of different spin are obtained by the 6th order WKB formula, and compared with the numerical values and 3th order WKB values. Second, motivated by the above reasons, we apply the obtained WKB formula for finding of the scalar quasinormal modes of multi-dimensional Schwarzshild black hole (Sec. III). It proves out that the real parts of the quasinormal frequencies are proportional to the product Dr_0^{-1}, where r_0 is the horizon radius, and D is the dimension of space-time.

2 Sixth order WKB analysis

First semi-analytical method for calculations of BH QNMs was apparently proposed by Bahram Mashhoon who used the Poschl-Teller potential to estimate the QN frequencies [24]. In [25] there was proposed a semi-analytical method for computing QNM's based on the WKB treatment. Then in [26] the first WKB order formula was extended to the third order beyond the eikonal approximation, and, afterwards, was frequently used in a lot of works (see for example [9] [28], [29], [30], [31], [32], [33], [34] and references therein).

The accuracy of the 3th order WKB formula (see eq. (1.5) in [26]) is the better, the more multipole number l and the less overtone n. For the Schwarzshild BH the results practically coincide with accurate numerical results of Leaver [35] at $l \geq 4$ when being restricted by lower overtones for which $l > n$. For fewer multipoles, however, accuracy is worse, and may reaches 10 per cents at $l = 0$, $n = 0$. Numerical approach [35], on contrary, is very accurate, but, dealing with numerical integration or systems of recurrence relations, is very cumbersome, and, often, require modification to be applied to different effective potentials. At the same time WKB approach lets us to obtain QNM's for a full range of parameters giving thereby some fields of work for intuition as to physical behavior of a system. Even though WKB formula gives the best accuracy at $l > n$, it includes the case of astrophysical black hole radiation where only lower overtones are significantly excited [36]. Both advantages and deficiencies of the WKB approach motivated us to extend the existent 3th order WKB formula up to the 6th order.

The perturbation equations of a black hole can be reduced to the Schrodinger wave-like equation:

$$\frac{d^2\psi}{dx^2} + Q(x)\psi(x) = 0, \tag{1}$$

where "the potential" $-Q(x)$ is constant at the event horizon ($x = -\infty$) and at the infinity ($x = +\infty$) and it rises to maximum at some intermediate $x = x_0$. Consider radiation of a given frequency ω incident on the black hole from infinity and let $R(\omega)$ and $T(\omega)$ be the reflection and transmission amplitudes respectively. Extend $R(\omega)$ to the complex frequency plane such that $Re(z) \neq 0$, and $T(z)/R(z)$ is regular. Then, the quasinormal modes correspond to the singularities of $R(z)$. We have a direct analogy with the problem of scattering near the pick of the potential barrier in quantum mechanics, where ω^2 plays a role of energy, and the two turning points divide the space into three regions at which boundaries the corresponding solutions should be matched.

To extend the 3th order WKB formula of [26] we used the technique of Iyer and Will. We shall omit here the technicalities of this approach which are described in [26]. The only thing we should stress is that since the coefficients M_{ij}, that connect amplitudes near the horizon with those at infinity, depend only on ν (related to the overtone number n) they may be found to higher orders, simply by solving the interior (between the turning points) problem to higher orders. Thus there is no need to perform an explicit match of the solutions to WKB solutions in the exterior (outside turning points) regions to the same order. The result has the form:

$$\frac{\imath Q_0}{\sqrt{2Q_0''}} - \Lambda_2 - \Lambda_3 - \Lambda_4 - \Lambda_5 - \Lambda_6 = n + \frac{1}{2}, \tag{2}$$

where the correction terms Λ_4, Λ_5, Λ_6 can be found in the Appendix I. Note that Λ_4 coincides with preliminary formula (A3) of [26] in proper designations.

An alternative, pure algebraic approach to finding higher order WKB corrections was proposed by O.Zaslavskii [37], using a quantum anharmonic oscillator problem where WKB correction terms come from perturbation theory corrections to the potential anharmonicity.

Thus we have obtained an economic and accurate formula for straightforward calculation of QNM frequencies. The 6th order formula applied to the $D = 4$ Schwarzshild BH is as accurate already at $l = 1$ as the 3th order formula does at $l = 4$. We show it in

Appendix II on example of QNM's corresponding to perturbations of fields of different spins: scalar ($s = 0$), neutrino ($s = 1/2$), electromagnetic ($s = 1$), gravitino ($s = 3/2$), and gravitational ($s = 2$). In addition, looking at the convergence of all sixth WKB values to some unknown true QN mode, we can judge, approximately, how far from the true QN value we are, staying within the framework of WKB method.

3 Quasinormal modes of the D-dimensional Schwarzshild black hole

The metric of the Schwarzshild black hole in D-dimensions has the form:

$$ds^2 = f(f)dt^2 - f^{-1}(r)dr^2 + r^2 d\Omega_{D-2}^2, \tag{3}$$

where

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^{D-3} = 1 - \frac{16\pi GM}{(D-2)\Omega_{D-2} r^{D-3}}. \tag{4}$$

Here we used the quantities

$$\Omega_{D-2} = \frac{(2\pi)^{(D-1)/2}}{\Gamma((D-1)/2)}, \quad \Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(z+1) = z\Gamma(z)$$

.

The scalar perturbation equation of this black hole can be reduced to the Schrodinger wave-like equation (1) with respect to the "tortoise" coordinate x: $dx = \frac{dr}{f(r)}$ where "the potential" $-Q(x)$ has the form:

$$Q(x) = \omega^2 - f(r)\left(\frac{l(l+D-3)}{r^2} + \frac{(D-2)(D-4)}{4r^2} f(r) + \frac{D-2}{2r} f'(r)\right), \tag{5}$$

At some fixed D we can put $r_0 = 2$ and measure ω in units $2r_0^{-1}$. The quasinormal modes satisfy the boundary conditions:

$$\phi(x) \sim c_{\pm} e^{\pm i\omega x} \qquad as \qquad x \to \pm\infty. \tag{6}$$

The 6th WKB order formula used here gives very accurate results for low overtones. The previous orders serve us to see the convergence of the WKB values of ω^2 as a WKB order grows to an accurate numerical result. Namely we can observe that for $l = 1, 2, 3, 4, ..$ for the fundamental overtone the 6th order values differs form its 5th order value by fractions of a percent or less at not very large D (we are restricted here by $D = 4, 5, ...15$).

It proves out that if one takes $r_0 = 2$ for each given D, then the real parts of ω for different D lay on a strict line. That is, ω_{Re} is proportional to the product $r_0 D$ (Remember that r_0 depends on D itself). Namely, for the fundamental overtone we obtain the following approximate relations:

$$\omega_{Re} \sim 0.244 D (r_0/2)^{-1}, \qquad l = 2 \tag{7}$$

$$\omega_{Re} \sim 0.275 D (r_0/2)^{-1}, \qquad l = 3 \tag{8}$$

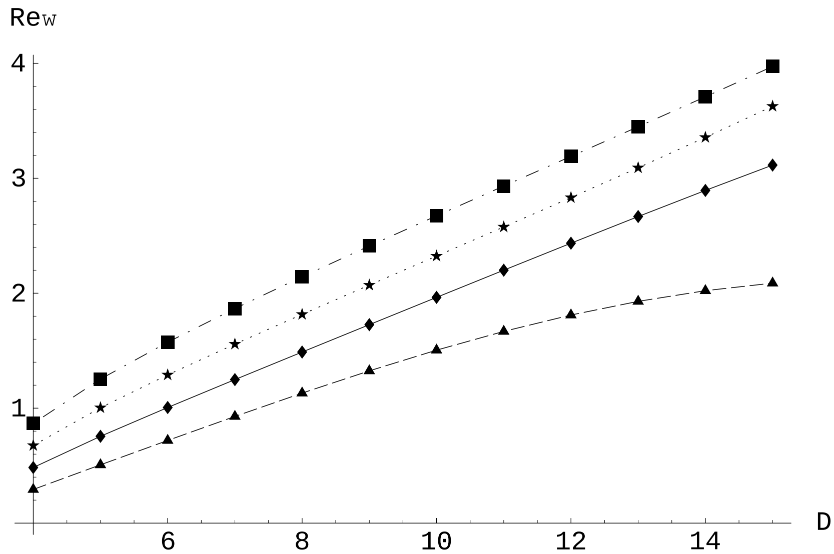

Figure 1: $Re\omega$ for different dimensions D; $l = 1$ (bottom), 2, 3, 4 (top); $n = 0$.

$$\omega_{Re} \sim 0.290 D (r_0/2)^{-1}, \qquad l = 4. \tag{9}$$

Here we take $\omega = \omega_{Re} - i\omega_{Im}$. Generally, the more the multipole number l, the more the coefficient before the product Dr_0^{-1}. The same $\sim Dr_0^{-1}$ relation we observed for higher overtone but not higher than l, for which WKB treatment is applicable. In Fig.1,2 we presented the real and imaginary parts of ω measured in $2r_0^{-1}$ for different D. For real parts of $l = 1$ modes we see the deviation from the strict line at large D. This however, is stipulated by a bad accuracy of the WKB approach, and we believe that the true frequencies will lay on strict line again. Indeed, one can judge about it by looking at the convergence plot Fig.3-Fig.6 where the real and imaginary parts of ω are shown as a function of the WKB order. Generally the accuracy of the WKB formula is the better, the more l, and the less n and D. Note that the dependence Dr_0^{-1} for lower overtones can be recovered even within 3th order formula, provided l is greater than 2, and D is not very large.

Another point is the $l = 0$ modes: in this case the lowest overtone implies $l = n$, and the WKB formula has considerable relative error. For a four-dimensional BH, for which the accurate numerical results are known, the error is about 10 percent for ω_{Im}, and 5 percent for ω_{Re} in the third WKB order, while in the sixth order it reduces to 0 percent for ω_{Re} and 3 percent for ω_{Im} (see Appendix II). For greater D the error increases, the difference between the fifth and sixth order WKB values grows and one cannot judge of true quasinormal behavior in this case (see Fig.3, 4, 5.). Fortunately, other field perturbations, including gravitational, have the lowest overtone with $l > n$ and the WKB treatment is of good accuracy for all l. In Table 1. we compare the third order WKB values of $l = 0$, $n = 0$ modes for different D [3] with those obtained through the sixth order here.

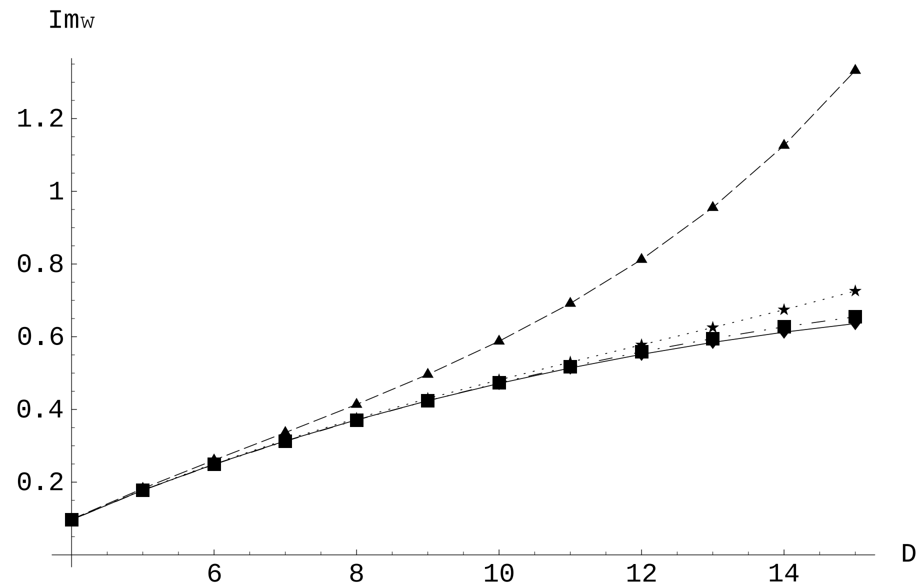

Figure 2: $Im\omega$ for different dimensions D; $l = 1$ (bottom), 2, 3, 4 (top); $n = 0$.

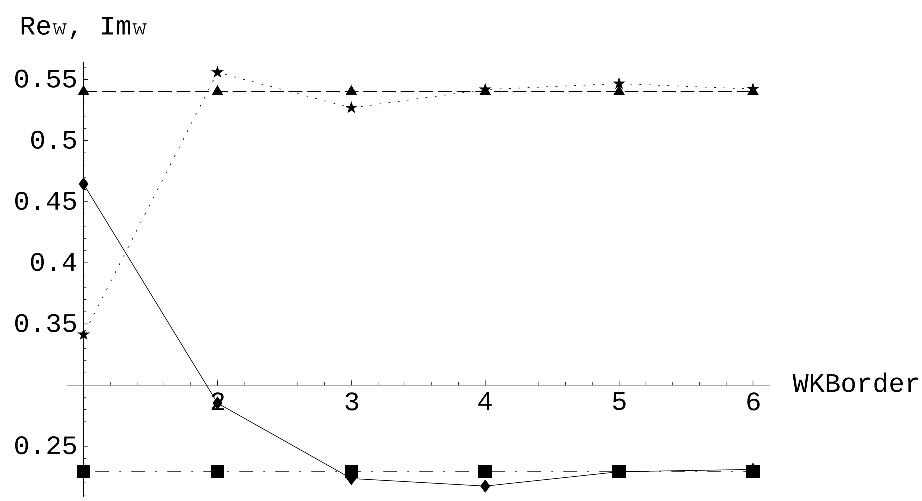

Figure 3: ω_{Re} (bottom) and ω_{Im} (top) as a function of WKB order of the formula with which it was obtained for $l = 1$, $n = 2$, $D = 4$ modes, and the corresponding numerical value. We see how the WKB values converge to an accurate numerical value as the WKB order increases.

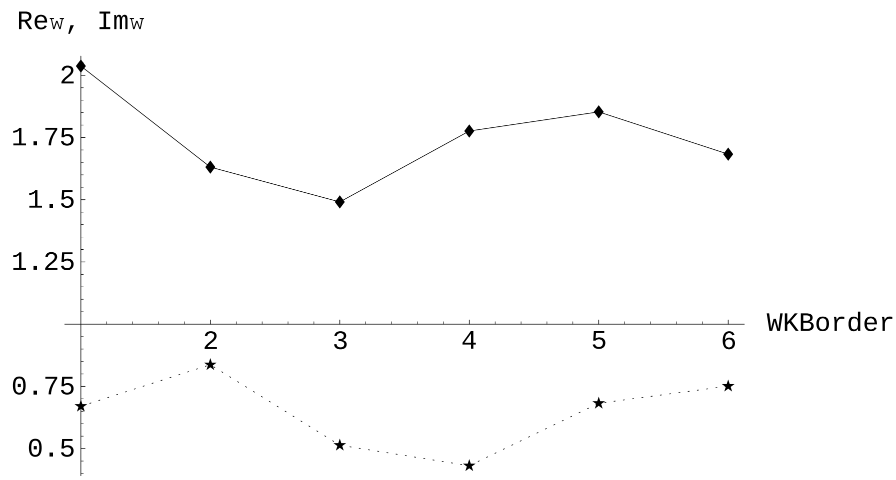

Figure 4: ω_{Re} (top) and ω_{Im} (bottom) as a function of WKB order of the formula with which it was obtained for $l = 0$, $n = 0$, $D = 12$ modes.

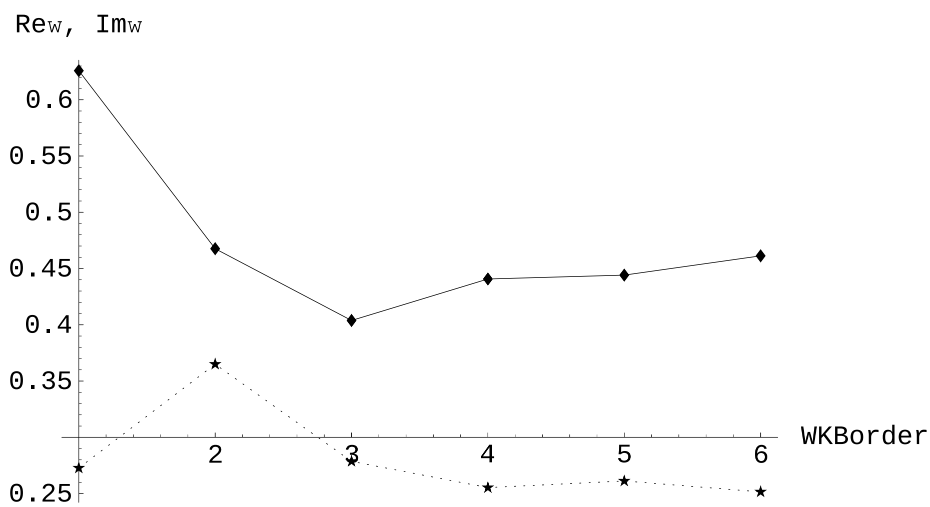

Figure 5: ω_{Re} (top) and ω_{Im} (bottom) as a function of WKB order of the formula with which it was obtained for $l = 0$, $n = 0$, $D = 6$ modes.

D	3th WKB order	6th WKB order	$1/r_0$
4	$0.1046 - 0.1152i$	$0.1105 - 0.1008i$	0.5
6	$1.0338 - 0.7133i$	$1.1808 - 0.6438i$	1.28
8	$1.9745 - 1.0258i$	$2.3004 - 1.0328i$	1.32
10	$2.7828 - 1.1596i$	$3.2214 - 1.3766i$	1.25
12	$3.4892 - 1.2020i$	$3.9384 - 1.7574i$	1.17

Table I. Schwarzschild QN frequencies for $l = 0$, $n = 0$ scalar perturbations in various D.

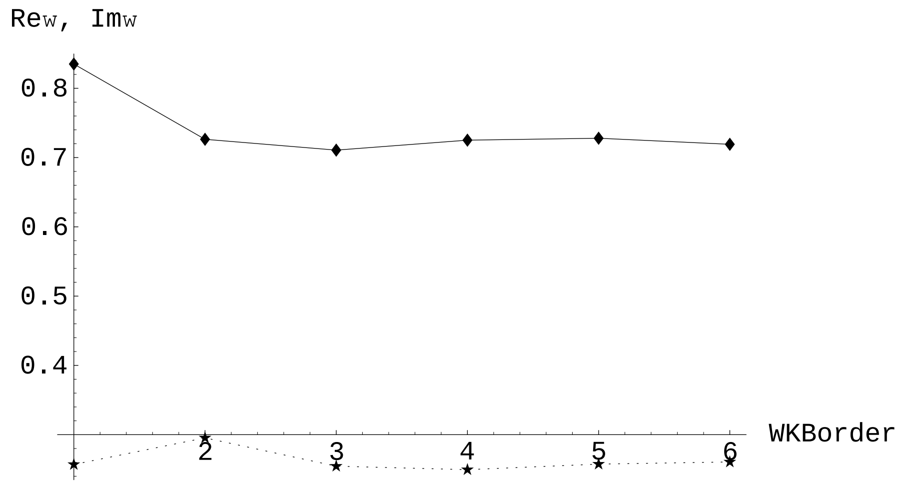

Figure 6: ω_{Re} (top) and ω_{Im} (bottom) as a function of WKB order of the formula with which it was obtained for $l = 1$, $n = 0$, $D = 6$ modes.

For large l the well-known approximate formula reads (see [38], [27], [39] for a proof)

$$\omega_{Re} = \frac{1}{3\sqrt{3}}\left(l + \frac{1}{2}\right), \qquad \omega_{Im} = \frac{1}{3\sqrt{3}}\left(n + \frac{1}{2}\right) \tag{10}$$

To obtain its D-dimensional generalization we find a value r_{max} at which the effective potential V attains its maximum, provided l is large

$$r_{max} \approx 2^{\frac{D-4}{D-3}}(D-1)^{\frac{1}{D-3}}, \qquad D = 4, 5, 6, \ldots. \tag{11}$$

Then let us make use of this value r_{max} when dealing with the first order WKB formula. After expansion in terms of small values of $1/l$, for a fixed D in units of $2r_0^{-1}$ we obtain

$$\omega_{Re} \approx \frac{D + 2l - 3}{4}\left(\frac{2}{D-1}\right)^{\frac{1}{D-3}}\sqrt{\frac{D-3}{D-1}} \tag{12}$$

$$\omega_{Im} \approx \frac{(D-3)}{4}\left(\frac{2}{D-1}\right)^{\frac{1}{D-3}}\frac{2n+1}{\sqrt{D-1}} \tag{13}$$

When $D = 4$ these formulas go over into (11). We see that when l is much larger than D, the $\sim Dr_0^{-1}$ dependence of ω_{Re} breaks down.

4 Conclusion

We were interested here in a question how dimensionality effects on quasinormal behavior of black holes. Yet, several interesting points are left beyond our consideration of low laying quasinormal modes of multi-dimensional black holes. First of all, one would like to understand the origin of the relation $\sim Dr_0^{-1}$ in ω_{Re} dependence. In this question it

is possible to try to explain it from the interpretations of QN modes as Breit-Wigner type resonances generated by a family of surfaces wave propagating close to the unstable circular photon orbit [40]. Second, we do not know whether $\sim Dr_0^{-1}$ dependence will be present for perturbations of other fields, and for more general backgrounds, such as multi-dimensional Reissner-Nordstrom or Kerr. We hope further investigations will clarify these points.

Acknowledgements

It is a pleasure to acknowledge stimulating discussions with Vitor Cardoso and Oleg Zaslavskii.

5 Appendix 1: Correction terms for WKB formula

Here we shall follow the designations: Q_0 means the value of the potential Q at its pick, while Q_i is the ith derivative of Q with respect to the tortoise coordinate x. Then Q_i^j is the jth power of the ith derivative of Q.

$$\Lambda_4 = \frac{1}{597196800\sqrt{2}Q_2^7\sqrt{Q_2}}(2536975Q_3^6 - 9886275Q_2Q_3^4Q_4 + 5319720Q_2^2Q_3^3Q_5 -$$
$$225Q_2^2Q_3^2(-40261Q_4^2 + 9688Q_2Q_6) + 3240Q_2^3Q_3(-1889Q_4Q_5 + 220Q_2Q_7) -$$
$$729Q_2^3(1425Q_4^3 - 1400Q_2Q_4Q_6 + 8Q_2(-123Q_5^2 + 25Q_2Q_8))) +$$
$$\frac{(n+1/2)^2}{4976640\sqrt{2}Q_2^7\sqrt{Q_2}}(348425Q_3^6 - 1199925Q_2Q_3^4Q_4 + 57276Q_2^2Q_3^3Q_5 -$$
$$45Q_2^2Q_3^2(-20671Q_4^2 + 4552Q_2Q_6) + 1980Q_2^3Q_3(-489Q_4Q_5 + 52Q_2Q_7) -$$
$$27Q_2^3(2845Q_4^3 - 2360Q_2Q_4Q_6 + 56Q_2(-31Q_5^2 + 5Q_2Q_8))) +$$
$$\frac{(n+1/2)^4}{2488320\sqrt{2}Q_2^7\sqrt{Q_2}}(192925Q_3^6 - 581625Q_2Q_3^4Q_4 + 234360Q_2^2Q_3^3Q_5 -$$
$$45Q_2^2Q_3^2(-8315Q_4^2 + 1448Q_2Q_6) + 1080Q_2^3Q_3(-161Q_4Q_5 + 12Q_2Q_7) -$$
$$27Q_2^3(625Q_4^3 - 440Q_2Q_4Q_6 + 8Q_2(-63Q_5^2 + 5Q_2Q_8))) \tag{14}$$

$$\Lambda_5 = \frac{(n+1/2)}{57330892800Q_2^{10}}(2768256Q_{10}Q_2^7 - 1078694575Q_3^8 + 5357454900Q_2Q_3^6Q_4 -$$
$$2768587920Q_2^2Q_3^5Q_5 + 90Q_2^2Q_3^4(-88333625Q_4^2 + 12760664Q_2Q_6) -$$
$$4320Q_2^3Q_3^3(-1451425Q_4Q_5 + 91928Q_2Q_7) - 27Q_2^4(7628525Q_4^4 - 9382480Q_2Q_4^2Q_6 +$$
$$64Q_2^2(19277Q_6^2 + 37764Q_5Q_7) + 576Q_2Q_4(-21577Q_5^2 + 2505Q_2Q_8)) +$$
$$540Q_2^3Q_3^2(6515475Q_4^3 - 3324792Q_2Q_4Q_6 + 16Q_2(-126468Q_5^2 + 12679Q_2Q_8)) -$$
$$432Q_2^4Q_3(5597075Q_4^2Q_5 - 854160Q_2Q_4Q_7 + 8Q_2(-145417Q_5Q_6 + 6685Q_2Q_9))) +$$

$$\frac{(n+1/2)^3}{477757440Q_2^{10}}(31104Q_{10}Q_2^7 - 42944825Q_3^8 + 193106700Q_2Q_3^6Q_4 -$$
$$-90039120Q_2^2Q_3^5Q_5 + 30Q_2^2Q_3^4(-8476205Q_4^2 + 1102568Q_2Q_6) -$$
$$4320Q_2^3Q_3^3(-41165Q_4Q_5 + 2312Q_2Q_7) - 9Q_2^4(445825Q_4^4 - 472880Q_2Q_4^2Q_6 +$$
$$64Q_2^2(829Q_6^2 + 1836Q_5Q_7) + 4032Q_2Q_4(-179Q_5^2 + 15Q_2Q_8)) +$$
$$180Q_2^3Q_3^2(532615Q_4^3 - 241224Q_2Q_4Q_6 + 16Q_2(-9352Q_5^2 + 799Q_2Q_8)) -$$
$$144Q_2^4Q_3(392325Q_4^2Q_5 - 51600Q_2Q_4Q_7 + 8Q_2(-8853Q_5Q_6 + 335Q_2Q_9))) +$$
$$\frac{(n+1/2)^5}{1194393600Q_2^{10}}(10368Q_{10}Q_2^7 - 66578225Q_3^8 + 272124300Q_2Q_3^6Q_4 -$$
$$112336560Q_2^2Q_3^5Q_5 + 9450Q_2^2Q_3^4(-33775Q_4^2 + 3656Q_2Q_6) -$$
$$151200Q_2^3Q_3^3(-1297Q_4Q_5 + 56Q_2Q_7) - 27Q_2^4(89075Q_4^4 - 83440Q_2Q_4^2Q_6 +$$
$$64Q_2^2(131Q_6^2 + 396Q_5Q_7) + 576Q_2Q_4(-343Q_5^2 + 15Q_2Q_8)) +$$
$$540Q_2^3Q_3^2(188125Q_4^3 - 71400Q_2Q_4Q_6 + 16Q_2(-3052Q_5^2 + 177Q_2Q_8)) -$$
$$432Q_2^4Q_3(118825Q_4^2Q_5 - 11760Q_2Q_4Q_7 + 8Q_2(-2303Q_5Q_6 + 55Q_2Q_9))) \tag{15}$$
$$\Lambda_6 = \frac{-i}{202263389798400Q_2^{12}\sqrt{2Q_2}}(-171460800Q_{12}Q_2^9 + 1714608000Q_{11}Q_2^8Q_3 -$$
$$10268596800Q_{10}Q_2^7Q_3^2 + 970010662775Q_3^{10} + 3772137600Q_{10}Q_2^8Q_4 -$$
$$62626341755525Q_2Q_3^8Q_4 + 137829831961500Q_2^2Q_3^6Q_4^2 - 119541481258500Q_2^3Q_3^4Q_4^3 +$$
$$3449170577475Q_2^4Q_3^2Q_4^4 - 144528059025Q_2^5Q_4^5 + 3352602187200Q_2^2Q_3^7Q_5 -$$
$$12300730092000Q_2^3Q_3^5Q_4Q_5 + 11994129604800Q_2^4Q_3^3Q_4^2Q_5 - 2624788605600Q_2^5Q_3Q_4^3Q_5 +$$
$$2580769643760Q_2^4Q_3^4Q_5^2 - 3453909784416Q_2^5Q_3^2Q_4Q_5^2 + 438440697072Q_2^6Q_4^2Q_5^2 +$$
$$+260524397952Q_2^6Q_3Q_5^3 - 1475306441280Q_2^3Q_3^6Q_6 + 4329682610400Q_2^4Q_3^4Q_4Q_6 -$$
$$2865128172480Q_2^5Q_3^2Q_4^2Q_6 + 233443879200Q_2^6Q_4^3Q_6 - 1660199804928Q_2^5Q_3^3Q_5Q_6 +$$
$$1281705296256Q_2^6Q_3Q_4Q_5Q_6 - 87403857408Q_2^7Q_5^2Q_6 + 231105873600Q_2^6Q_3^2Q_6^2 -$$
$$68412859200Q_2^7Q_4Q_6^2 + 552968700480Q_2^4Q_3^5Q_7 - 1231789749120Q_2^5Q_3^3Q_4Q_7 +$$
$$470726303040Q_2^6Q_3Q_4^2Q_7 + 413953400448Q_2^6Q_3^2Q_5Q_7 - 126242178048Q_2^7Q_4Q_5Q_7 -$$
$$91489305600Q_2^7Q_3Q_6Q_7 + 5619715200Q_2^8Q_7^2 - 175752294480Q_2^5Q_3^4Q_8 +$$
$$271759652640Q_2^6Q_3^2Q_4Q_8 - 39736040400Q_2^7Q_4^2Q_8 - 73378363968Q_2^7Q_3Q_5Q_8 +$$
$$9773265600Q_2^8Q_6Q_8 + 47107126080Q_2^6Q_3^3Q_9 - 43345290240Q_2^7Q_3Q_4Q_9 + 7400248128Q_2^8Q_5Q_9) -$$
$$\frac{(n+1/2)^2 i}{687970713600Q_2^{12}\sqrt{2Q_2}}(-4551552Q_{12}Q_2^9 + 60279552Q_{11}Q_2^8Q_3 -$$
$$425036160Q_{10}Q_2^7Q_3^2 + 73727194625Q_3^{10} + 116743680Q_{10}Q_2^8Q_4 -$$
$$443649208275Q_2Q_3^8Q_4 + 901144103850Q_2^2Q_3^6Q_4^2 - 711096726150Q_2^3Q_3^4Q_4^3 +$$

$$182164306725Q_2^4Q_3^2Q_4^4 - 6289615575Q_2^5Q_4^5 + 222467624400Q_2^2Q_3^7Q_5 -$$
$$746418445200Q_2^3Q_3^5Q_4Q_5 + 653423900400Q_2^4Q_3^3Q_4^2Q_5 - 124319674800Q_2^5Q_3Q_4^3Q_5 +$$
$$143980943040Q_2^4Q_3^4Q_5^2 - 169712521920Q_2^5Q_3^2Q_4Q_5^2 + 18188188416Q_2^6Q_4^2Q_5^2 +$$
$$11240861184Q_2^6Q_3Q_5^3 - 91198200240Q_2^3Q_3^6Q_6 + 241513732080Q_2^4Q_3^4Q_4Q_6 -$$
$$140030897040Q_2^5Q_3^2Q_4^2Q_6 + 9200103120Q_2^6Q_4^3Q_6 - 84218693760Q_2^5Q_3^3Q_5Q_6 +$$
$$55248386688Q_2^6Q_3Q_4Q_5Q_6 - 3173043456Q_2^7Q_5^2Q_6 + 10464952896Q_2^6Q_3^2Q_6^2 -$$
$$2403421632Q_2^7Q_4Q_6^2 + 31637744640Q_2^4Q_3^5Q_7 - 62649953280Q_2^5Q_3^3Q_4Q_7 +$$
$$20409822720Q_2^6Q_3Q_4^2Q_7 + 18860532480Q_2^6Q_3^2Q_5Q_7 - 4693344768Q_2^7Q_4Q_5Q_7 -$$
$$3625731072Q_2^7Q_3Q_6Q_7 + 188054784Q_2^8Q_7^2 - 9155635200Q_2^5Q_3^4Q_8 +$$
$$12238024320Q_2^6Q_3^2Q_4Q_8 - 1405278720Q_2^7Q_4^2Q_8 - 2866700160Q_2^7Q_3Q_5Q_8 +$$
$$303295104Q_2^8Q_6Q_8 + 2210705280Q_2^6Q_3^3Q_9 - 1685525760Q_2^7Q_3Q_4Q_9 + 235488384Q_2^8Q_5Q_9) -$$

$$\frac{(n+1/2)^4 i}{20065812480Q_2^{12}\sqrt{2Q_2}}(-66528Q_{12}Q_2^9 + 1245888Q_{11}Q_2^8Q_3 -$$
$$11158560Q_{10}Q_2^7Q_3^2 + 4668804525Q_3^{10} + 2116800Q_{10}Q_2^8Q_4 -$$
$$25898331375Q_2Q_3^8Q_4 + 47959232650Q_2^2Q_3^6Q_4^2 - 33861927750Q_2^3Q_3^4Q_4^3 +$$
$$7454763225Q_2^4Q_3^2Q_4^4 - 184988475Q_2^5Q_4^5 + 11891917800Q_2^2Q_3^7Q_5 -$$
$$36105463800Q_2^3Q_3^5Q_4Q_5 + 27953667000Q_2^4Q_3^3Q_4^2Q_5 - 4457716200Q_2^5Q_3Q_4^3Q_5 +$$
$$6285855240Q_2^4Q_3^4Q_5^2 - 6471756144Q_2^5Q_3^2Q_4Q_5^2 + 565259688Q_2^6Q_4^2Q_5^2 +$$
$$380939328Q_2^6Q_3Q_5^3 - 4375251160Q_2^3Q_3^6Q_6 + 10317018600Q_2^4Q_3^4Q_4Q_6 -$$
$$5113813320Q_2^5Q_3^2Q_4^2Q_6 + 238888440Q_2^6Q_4^3Q_6 - 3203871552Q_2^5Q_3^3Q_5Q_6 +$$
$$1758685824Q_2^6Q_3Q_4Q_5Q_6 - 88566912Q_2^7Q_5^2Q_6 + 335466432Q_2^6Q_3^2Q_6^2 -$$
$$55073088Q_2^7Q_4Q_6^2 + 1351294560Q_2^4Q_3^5Q_7 - 2341442880Q_2^5Q_3^3Q_4Q_7 +$$
$$626542560Q_2^6Q_3Q_4^2Q_7 + 619520832Q_2^6Q_3^2Q_5Q_7 - 123524352Q_2^7Q_4Q_5Q_7 -$$
$$965744464Q_2^7Q_3Q_6Q_7 + 4048704Q_2^8Q_7^2 - 341160120Q_2^5Q_3^4Q_8 +$$
$$386210160Q_2^6Q_3^2Q_4Q_8 - 30837240Q_2^7Q_4^2Q_8 - 78073632Q_2^7Q_3Q_5Q_8 +$$
$$5848416Q_2^8Q_6Q_8 + 70415520Q_2^6Q_3^3Q_9 - 43424640Q_2^7Q_3Q_4Q_9 + 5255712Q_2^8Q_5Q_9) -$$

$$\frac{(n+1/2)^6 i}{300987187200Q_2^{12}\sqrt{2Q_2}}(-72576Q_{12}Q_2^9 + 1886976Q_{11}Q_2^8Q_3 -$$
$$22135680Q_{10}Q_2^7Q_3^2 + 27463538375Q_3^{10} + 2903040Q_{10}Q_2^8Q_4 -$$
$$141448688325Q_2Q_3^8Q_4 + 240655765350Q_2^2Q_3^6Q_4^2 - 152907158250Q_2^3Q_3^4Q_4^3 +$$
$$28724479875Q_2^4Q_3^2Q_4^4 - 413669025Q_2^5Q_4^5 + 59058073200Q_2^2Q_3^7Q_5 -$$
$$164264209200Q_2^3Q_3^5Q_4Q_5 + 113654696400Q_2^4Q_3^3Q_4^2Q_5 - 15166342800Q_2^5Q_3Q_4^3Q_5 +$$

$$
\begin{gathered}
26061194880Q_2^4Q_3^4Q_5^2 - 23876233920Q_2^5Q_3^2Q_4Q_5^2 + 1767189312Q_2^6Q_4^2Q_5^2 + \\
1292433408Q_2^6Q_3Q_5^3 - 18902165520Q_2^3Q_3^6Q_6 + 40256773200Q_2^4Q_3^4Q_4Q_6 - \\
17116974000Q_2^5Q_3^2Q_4^2Q_6 + 483582960Q_2^6Q_4^3Q_6 - 11384150400Q_2^5Q_3^3Q_5Q_6 + \\
5285056896Q_2^6Q_3Q_4Q_5Q_6 - 246903552Q_2^7Q_5^2Q_6 + 992779200Q_2^6Q_3^2Q_6^2 - \\
101860416Q_2^7Q_4Q_6^2 + 4966859520Q_2^4Q_3^5Q_7 - 7661606400Q_2^5Q_3^3Q_4Q_7 + \\
1683037440Q_2^6Q_3Q_4^2Q_7 + 1861574400Q_2^6Q_3^2Q_5Q_7 - 316141056Q_2^7Q_4Q_5Q_7 - \\
235146240Q_2^7Q_3Q_6Q_7 + 8895744Q_2^8Q_7^2 - 1042372800Q_2^5Q_3^4Q_8 + \\
1016789760Q_2^6Q_3^2Q_4Q_8 - 52436160Q_2^7Q_4^2Q_8 - 189060480Q_2^7Q_3Q_5Q_8 + \\
9217152Q_2^8Q_6Q_8 + 175190400Q_2^6Q_3^3Q_9 - 87816960Q_2^7Q_3Q_4Q_9 + 10378368Q_2^8Q_5Q_9) \quad (16)
\end{gathered}
$$

All six WKB corrections printed in MATEMATICA are available from the author in electronic form upon request.

6 Appendix 2: QNMs of a 4-dimensional Schwarzshild black hole

"The potential" $Q(x)$ in case of a Schwarzshild black hole has the form

$$
Q(x) = \omega^2 - \left(1 - \frac{1}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{1-s^2}{r^3}\right), \tag{17}
$$

where $s = 0$ corresponds to scalar perturbations, $s = 1/2$ - neutrino perturbations, $s = 1$ - electromagnetic perturbations, $s = 3/2$ - gravitino perturbations, $s = 2$ - gravitational perturbations. The quasinormal frequencies at 3th and 6th WKB orders and in comparison with numerical results [35] are presented in the table I.

$s = 0$	numerical	3th order WKB	6th order WKB
$l = 0,\ n = 0$	$0.1105 - 0.1049i$	$0.1046 - 0.1152i$	$0.1105 - 0.1008i$
$l = 1,\ n = 0$	$0.2929 - 0.0977i$	$0.2911 - 0.0980i$	$0.2929 - 0.0978i$
$l = 1,\ n = 1$	$0.2645 - 0.3063i$	$0.2622 - 0.3074i$	$0.2645 - 0.3065i$
$l = 2,\ n = 0$	$0.4836 - 0.0968i$	$0.4832 - 0.0968i$	$0.4836 - 0.0968i$
$l = 2,\ n = 1$	$0.4639 - 0.2956i$	$0.4632 - 0.2958i$	$0.4638 - 0.2956i$
$l = 2,\ n = 2$	$0.4305 - 0.5086i$	$0.4317 - 0.5034i$	$0.4304 - 0.5087i$

$s = 1/2$	numerical	3th order WKB	6th order WKB
$l = 1,\ n = 0$	$-$	$0.2803 - 0.0969i$	$0.2822 - 0.0967i$
$l = 1,\ n = 1$	$-$	$0.2500 - 0.3049i$	$0.2525 - 0.3040i$
$l = 2,\ n = 0$	$-$	$0.4768 - 0.9639i$	$0.4772 - 0.0963i$
$l = 2,\ n = 1$	$-$	$0.4565 - 0.2947i$	$0.4571 - 0.2945i$
$l = 2,\ n = 2$	$-$	$0.4244 - 0.5016i$	$0.4231 - 0.5070i$
$l = 3,\ n = 0$	$-$	$0.6706 - 0.0963i$	$0.6708 - 0.0963i$
$l = 3,\ n = 1$	$-$	$0.6557 - 0.2917i$	$0.6560 - 0.2917i$
$l = 3,\ n = 2$	$-$	$0.6299 - 0.4931i$	$0.6286 - 0.4950i$
$l = 3,\ n = 3$	$-$	$0.5970 - 0.6997i$	$0.5932 - 0.7102i$

$s = 1$	numerical	3th order WKB	6th order WKB
$l = 1,\ n = 0$	$0.2483 - 0.0925i$	$0.2459 - 0.0931i$	$0.2482 - 0.0926i$
$l = 1,\ n = 1$	$0.2145 - 0.2937i$	$0.2113 - 0.2958i$	$0.2143 - 0.2941i$
$l = 2,\ n = 0$	$0.4576 - 0.0950i$	$0.4571 - 0.0951i$	$0.4576 - 0.0950i$
$l = 2,\ n = 1$	$0.4365 - 0.2907i$	$0.4358 - 0.2910i$	$0.4365 - 0.2907i$
$l = 2,\ n = 2$	$0.4012 - 0.5016i$	$0.4023 - 0.4959i$	$0.4009 - 0.5017i$
$l = 3,\ n = 0$	$0.6569 - 0.0956i$	$0.6567 - 0.0956i$	$0.6569 - 0.0956i$
$l = 3,\ n = 1$	$0.6417 - 0.2897i$	$0.6415 - 0.2898i$	$0.6417 - 0.2897i$
$l = 3,\ n = 2$	$0.6138 - 0.4921i$	$0.6151 - 0.4901i$	$0.6138 - 0.4921i$
$l = 3,\ n = 3$	$0.5779 - 0.7063i$	$0.5814 - 0.6955i$	$0.5775 - 0.7065i$

$s = 3/2$	numerical	3th order WKB	6th order WKB
$l = 1,\ n = 0$	$-$	$0.1817 - 0.0866i$	$0.1739 - 0.08357i$
$l = 1,\ n = 1$	$-$	$0.1354 - 0.2812i$	$0.1198 - 0.2813i$
$l = 2,\ n = 0$	$-$	$0.4231 - 0.926i$	$0.4236 - 0.0925i$
$l = 2,\ n = 1$	$-$	$0.4000 - 0.2842i$	$0.4007 - 0.2838i$
$l = 2,\ n = 2$	$-$	$0.3636 - 0.4853i$	$0.3618 - 0.4919i$
$l = 3,\ n = 0$	$-$	$0.6332 - 0.0945i$	$0.6333 - 0.0944i$
$l = 3,\ n = 1$	$-$	$0.6173 - 0.2864i$	$0.6175 - 0.2863i$
$l = 3,\ n = 2$	$-$	$0.5898 - 0.4846i$	$0.5884 - 0.4868i$
$l = 3,\ n = 3$	$-$	$0.5547 - 0.6882i$	$0.5505 - 0.7000i$

$s = 2$	numerical	3th order WKB	6th order WKB
$l = 2,\ n = 0$	$0.3737 - 0.0890i$	$0.3732 - 0.0892i$	$0.3736 - 0.0890i$
$l = 2,\ n = 1$	$0.3467 - 0.2739i$	$0.3460 - 0.2749i$	$0.3463 - 0.2735i$
$l = 2,\ n = 2$	$0.3011 - 0.4783i$	$0.3029 - 0.4711i$	$0.2985 - 0.4776i$
$l = 3,\ n = 0$	$0.5994 - 0.0927i$	$0.5993 - 0.0927i$	$0.5994 - 0.0927i$
$l = 3,\ n = 1$	$0.5826 - 0.2813i$	$0.5824 - 0.2814i$	$0.5826 - 0.2813i$
$l = 3,\ n = 2$	$0.5517 - 0.4791i$	$0.5532 - 0.4767i$	$0.5516 - 0.4790i$
$l = 3,\ n = 3$	$0.5120 - 0.6903i$	$0.5157 - 0.6774i$	$0.5111 - 0.6905i$
$l = 4,\ n = 0$	$0.8092 - 0.0942i$	$0.8091 - 0.0942i$	$0.8092 - 0.0942i$
$l = 4,\ n = 1$	$0.7966 - 0.2843i$	$0.7965 - 0.2844i$	$0.7966 - 0.2843i$
$l = 4,\ n = 2$	$0.7727 - 0.4799i$	$0.7736 - 0.4790i$	$0.7727 - 0.4799i$
$l = 4,\ n = 3$	$0.7398 - 0.6839i$	$0.7433 - 0.6783i$	$0.7397 - 0.6839i$
$l = 4,\ n = 4$	$0.7015 - 0.8982i$	$0.7072 - 0.8813i$	$0.7006 - 0.8985i$

References

[1] S.Dimopoulos and S.Landsberg, Phys. Rev. Lett. **87** 161602 (2002)

[2] V.P.Frolov and D.Stojkovic, gr-qc 0301016

[3] V.Cardoso, O.J.C. Dias, and J.P.S. Lemos, hep-th 0212168

[4] G.T.Horowitz and V.Hubeny *Phys.Rev.* **D62** 024027 (2000)

[5] J.S.F.Chan and R.B.Mann *Phys.Rev.* **D59** 064025 (1999)

[6] V.Cardoso and J.P.S.Lemos *Phys.Rev.* **D63** 124015 (2001)

[7] D. Birmingham, I. Sachs, and S.N. Solodukhin, Phys. Rev. Lett. **88** 151301 (2002)

[8] D. Birmingham, I. Sachs, and S.N. Solodukhin, hep-th/0212308 (2002)

[9] R.A.Konoplya, *Phys.Rev.* **D66** 084007 (2002)

[10] R.A.Konoplya, *Phys.Rev.* **D66** 044009 (2002)

[11] A. O. Starinets, *Phys.Rev.* **D66** 124013 (2002).

[12] D. T. Son, A. O. Starinets, J.H.E.P. **0209**:042, (2002)

[13] R.Aros, C.Martinez, R.Troncoso, and J.Zanelli, hep-th/0211024;

[14] M.Musiri and G.Siopsis, hep-th/0301081.

[15] I.G.Moss and J.P.Norman, Class.Quant.Grav. **19**, 2323 (2002) (2002)

[16] K.Kokkotas and B.Schmidt, "Quasi-normal modes of stars and black holes", Living.Reviews.Relativ., **2**, 2 (1999); H. P. Nollert, Class. Quantum Grav. **16**, R159 (2000).

[17] O. Dreyer, gr-qc/0211076.

[18] G. Kunstatter, gr-qc/0211076.

[19] V.Cardoso and J.P.S.Lemos, gr-qc/0301078.

[20] L. Motl, gr-qc/0212096.

[21] W.T.Kim and J.J.Oh, *Phys.Lett.* **B514** 155 (2001)

[22] R.A.Konoplya, Phys.Lett. **B 550**, 117 (2002)

[23] L. Motl and A. Neitzke, hep-th/0301173;

[24] H-J.Blome and B.Mashhoon, Phys.Lett. **100A** 231 (1984)

[25] B.F.Schutz and C.M.Will *Astrophys.J.Lett* **291** L33 (1985)

[26] S.Iyer and C.M.Will, *Phys.Rev.* **D35** 3621 (1987)

[27] S.Iyer, *Phys.Rev.* **D35** 3632 (1987)

[28] K.Kokkotas, and B.F.Schutz, *Phys.Rev.* **D37** 3378 (1988)

[29] K.Kokkotas, *Nuovo.Cimento.* **B108** 991 (1993)

[30] L.E.Simone and C.M.Will, Class.Quant.Grav. **9**, 963 (1992)

[31] N.Andersson and H.Onozawa, *Phys.Rev.* **D54** 7470 (1996)

[32] V.Ferrari, M.Pauri and F.Piazza, *Phys.Rev.* **D63** 064009 (2001)

[33] H.Onozawa, T.Okamura, T.Mishima, and H.Ishihara, *Phys.Rev.* **D53** 7033 (1996)

[34] R.A.Konoplya, Gen.Relativ.Grav. **34**, 329 (2002)

[35] E.Leaver, Proc.R.Soc.London **A402**, 285 (1985)

[36] R.F.Stark and T.Piran, Phys. Rev. Lett. **55** 891 (1985)

[37] O.B.Zaslavskii, *Phys.Rev.* **D43** 605 (1991)

[38] V.Ferrari and B.Mashhoon, Phys. Rev. Lett. **52** 1361 (1984).

[39] W.H.Press, *Astrophys.J* **170** L105 (1971)

[40] N.Andersson, Class.Quant.Grav. **11**, 3003 (1994); Y.Decanini, A.Folacci and B.Jensen, gr-qc/0212093.